This will be recognized as being equal to the lateral stiffness of a beam without axial load, since the tip deflection y of such a beam under lateral load F at the tip is given by

$$y_t = FL^3/3EI \quad (6)$$

Using the beam-column stiffness $K(P)$ in place of K in the Timoshenko and Gere formula, Eq. (1) gives

$$P = \frac{P(CL)}{L}\,\frac{1}{(\tan\beta L/\beta L - 1)} \quad (7)$$

or

$$(\tan\beta L/\beta L) - 1 = C \quad (8)$$

which is exactly Parnes' Eq. (4). For small values of C, we have the buckling of the rigid link in rotation as the predominating failure mode. From Eqs. (1) and (5) of this Comment or by use of the series expansion Eq. (4) in Eq. (8), this load is

$$P \approx (3EI/L^3)CL \quad (9)$$

This illustrates the "paradoxical" behavior at small values of C noted by Parnes and is seen here to be essentially the same physical phenomenon dealt with by Timoshenko and Gere.

As C approaches infinity, buckling of the elastic column predominates, and from Eq. (8) the critical load is given by $\beta L = \pi/2$, which is the classical buckling load of the cantilever column. As Parnes showed, consideration of the finite stiffness of the link BC introduces no interaction with the phenomena just discussed but merely requires the additional consideration of a possible buckling condition of the link BC, considered to be simply supported at both ends.

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Reply by Author to A. H. Flax

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THE author wishes to thank Dr. Flax for his pertinent discussion of the apparent "paradox" and for bringing to his attention an explanation of the paradox by considering the lateral deflection of a rigid link resisted by a spring of effective stiffness $K(P)$. While this explanation is quite correct and ingenious, the author believes the explanation given in the previous discussion[1] to be more direct, as it derives immediately from considerations of statics.

Although it is well known that systems of rigid links connected in various ways by means of elastic springs often lead to increasing values of critical loads with member length, the author believes that this phenomenon had not been previous shown to exist for cases of elastic columns that undergo flexure.

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Comment on "Effects of Radial Appendage Flexibility on Shaft Whirl Stability"

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WILGEN and Schlack[1] investigate the stability of a rotating shaft with two opposite radial booms attached to it (see Fig. 1). They give the stability limits as a function of the flexibilities of the components and of other system parameters. In the following it is shown that 1) analytical solutions can be given for the shaft deflections which are approximated by a series in Ref. 1, 2) simple and accurate numerical solutions do exist for the deflections of the radial booms, which are approximated by one single "comparison function" in Ref. 1, and 3) the out-of-plane stability of the system is strongly dependent on the flexibility of the radial booms and not independent of that parameter as stated in Ref. 1. The in-plane stability limits given in Ref. 1 will be recalculated on the basis of the preceding comments and new stability limits will be given for the out-of-plane motion.

The existence of analytical solutions for rotating shafts is well known.[2] Less known perhaps is the fact that analytical solutions for rotating symmetric shafts can be derived from the well-known solutions for nonrotating shafts (i.e., booms) by pure coordinate transformation.[3] This is a direct consequence of the fact that a rotation with respect to the shaft axis is not reflected in the differential equations for a symmetric shaft if the system of reference is not affected by this rotation. In general, however (but not always), rotating coordinates are used with rotating shafts. That is why the spin appears in the equations. The fact that the differential equations (and consequently the dynamic behavior) of a rotating symmetric shaft are not affected by the speed of rotation is only true for conservative systems. Inner damping forces, for instance, do depend on the shaft's rotation, and it is only due to these forces that a shaft becomes unstable at a certain critical speed.

For radial booms, on the other hand, analytical solutions are not known because the underlying differential equations have nonconstant parameters arising from the nonconstant axial (centrifugal) force. However, numerical solutions of the differential equations can be given with a precision that depends only on computer accuracy and not on modal truncation or discretization effects. Here the boom deflections are expressed by a power series.

The recurrence law for the generation of the coefficients of the series follows directly from the coefficients of the dif-

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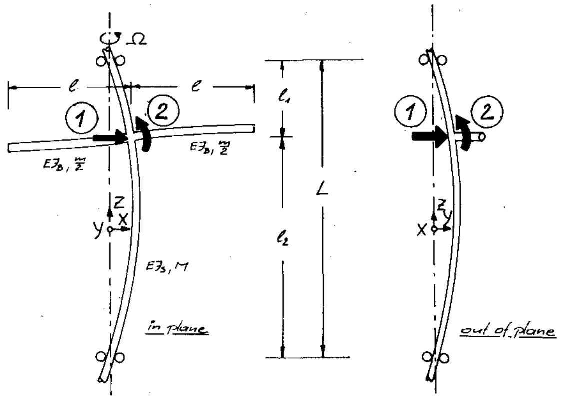

Fig. 1 Shaft boom configuration.

ferential equation. The method fails for very low bending stiffness, say

$$EI_B/[\Omega^2(m/2)\ell^3] < .001$$

(The nomenclature follows from Fig. 1, where M is the total shaft mass and m is the combined mass of both booms.) Then the device is a cable rather than a boom and consequently the differential equations for cables should be used from the outset. For these, the procedure just described can also be applied.

Boom deflections out of the plane formed by the shaft and the boom are not included in Ref. 1. Consequently, for out-of-plane motion the system is equivalent to a rotating boom with an attached mass point. The results presented here do include out-of-plane flexibility. It is interesting that the numerical procedure developed for the in-plane motion can be directly applied to the out-of-plane motion, since the related differential equation follows from the former if ω^2 is replaced by $\omega^2+\Omega^2$ systematically, where Ω is the spin velocity and ω is the frequency parameter.

Having solved the differential equations for the individual parts, we may write the dynamic equations for the complete system if the concept of dynamic stiffness is used. Unfortunately, this method is not very popular with mechanical engineers, although it offers a number of striking advantages over the methods in general use. The procedure to be followed is presented in Ref. 4, for instance. It has been adopted to rotating systems in Ref. 3. More details of the method are given in Ref. 5. With the rise of these methods, the complete dynamics are given in terms of a dynamic stiffness matrix related to the four degrees of freedom of the cross joint of shaft and booms. The system's stability follows from the positive definiteness of the dynamic stiffness matrix with $\omega=0$ (i.e., the static stiffness matrix). In the case under consideration, the 4×4 matrix splits up into two decoupled 2×2 matrices for in-plane and out-of-plane deformation; see Fig. 2. The vector of displacements (and forces) used in Fig. 2 is indicated in Fig. 1. The terms in the matrix which are due to the shaft are identical for in-plane and out-of-plane motion. Therefore, both matrices are described together in Fig. 2, the distinction between the two cases being self explanatory.

Evaluation of the stiffness matrices of Fig. 2 for the stability limits results in the stability diagrams given in Figs. 3-5. If $\ell_1=0$, the radial booms have no effect on the out of plane stability, i.e. the stability limit is given by $\Omega^2/\Omega_{cr}^2=1$.

Comparing Figs. 3 and 4 to the respective diagrams given in Ref. 1, we can state that there are no drastic differences. Nevertheless, it should be noted that the stable region is remarkably overestimated in Ref. 1 in some cases. At the far right of Figs. 3 and 4, where rigid radial booms are referred to, the difference is more than 10% in Ω^2/Ω_{cr}^2 for $m/M=1$. As the choice of the comparison function does not matter for nearly rigid booms, this difference is only due to truncation of the shaft modes higher than 10 in Ref. 1.

On the left-hand side of the diagrams (cable booms) predicted limiting values are approached both in Ref. 1 and here. This is better than one would expect since the comparison function used in Ref. 1 (quite deliberately, the first mode of the cantelever boom without centrifugal force) differs considerably from the true function. In Fig. 6 the comparison function and the true functions are plotted vs. $EI_B/[\Omega^2(m/2)\ell^3]$. They have been normalized to the same tip amplitude. For high bending stiffness the true function does not exactly approach the comparison function of Ref. 1, but the differences are within plotting accuracy.

The stability limits for out-of-plane motion are given in Fig. 5. The stability limits predicted in Ref. 1 for this case are only approached on the far right, i.e., when the stiffness approaches infinity.

For all other values of the stiffness parameters out-of-plane stability is lower, and, for zero bending stiffness (of the radial booms), the stable region (out-of-plane) reduces to zero, even for very small (but nonzero) boom masses.

In practice, the problem is not as severe since the radial booms are never fixed to the shaft axis at zero radius as assumed here. With an offset r the quantity Z_I in the stiffness matrix becomes ℓ/r for zero bending stiffness.[3] Only if r is reduced to zero will the destabilizing effect of the radial cable boom approach infinity and the stable region approach zero. What actually happens in this case is that the cable booms both turn to the same side. Because of nonlinear effects the system finds a new equilibrium position.

$2\lambda_1^3\cos\lambda_1\cosh\lambda_1/(D_1\ell_1^3)$ $+2\lambda_2^3\cos\lambda_2\cosh\lambda_2/(D_2\ell_2^3)$ $-m\Omega^2$ or $-m\Omega^2(1+Z_A)$ (in plane) (out of plane)	$-\lambda_1^2(\cos\lambda_1\sinh\lambda_1+\sin\lambda_1\cosh\lambda_1)/(D_1\ell_1^2)$ $+\lambda_2^2(\cos\lambda_2\sinh\lambda_2+\sin\lambda_2\cosh\lambda_2)/(D_2\ell_2^2)$
symmetric	$2\lambda_1\sin\lambda_1\sinh\lambda_1/(D_1\ell_1)$ $+2\lambda_2\sin\lambda_2\sinh\lambda_2/(D_2\ell_2)$ $+m\ell^2\Omega^2(1/3-Z_B)$ or 0 (in plane) (out of plane)

where $\lambda_i = \dfrac{\Omega^2\frac{M}{L}\ell_i^4}{EJ_s}$

$D_i = \dfrac{\sin\lambda_i\cosh\lambda_i-\cos\lambda_i\sinh\lambda_i}{EJ_s}$

$i=1,2$

Z_A and Z_B are defined in Ref. 3.

Fig. 2 Dynamic stiffness matrices for in-plane and out-of-plane deformations.

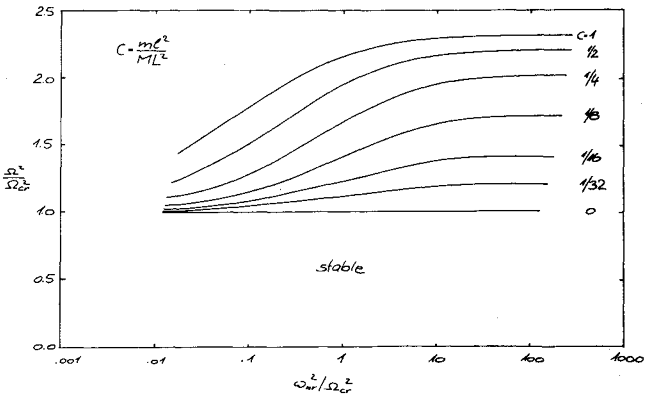

Fig. 3 Stability boundaries in plane for $\ell_2/L=0$.

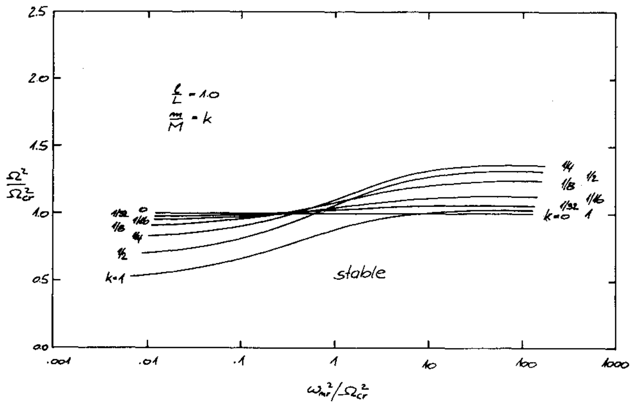

Fig. 4 Stability boundaries in plane for $\ell_2/L=0.75$.

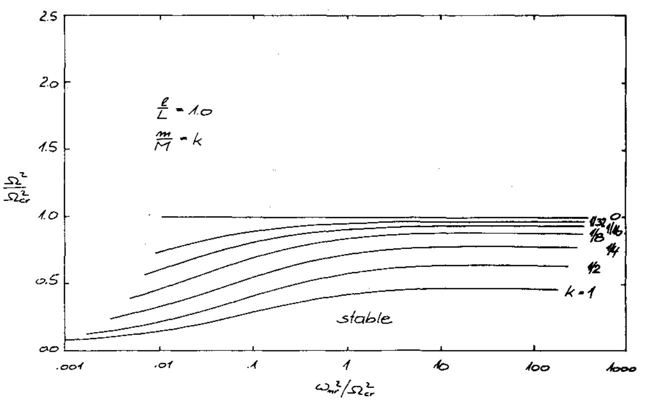

Fig. 5 Stability boundaries out of plane for $\ell_2/L=0.75$.

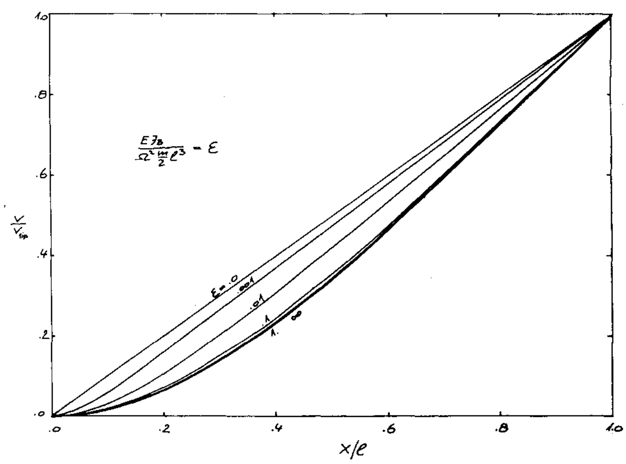

Fig. 6 Boom in-plane deflections for different boom stiffnesses.

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Reply by Authors to P. Kulla

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THE comments of Kulla are appreciated. They provide an important check on the results of Ref. 1 by an alternative method of analysis and provide further insight into the problem.

The observation that the stability boundaries should be somewhat lower than originally reported for shafts with very large rigid beam appendages is confirmed. Convergence rates are found to deteriorate for these cases so that ten terms may not be sufficient in this region. Inclusion of additional terms tends to lower these boundaries as reported by Kulla. Although the stability boundaries are generally much less sensitive to further refinement of the radial beam representation than to that of the shaft, particularly for the relatively rigid beams as noted by Kulla, inclusion of additional terms for the radial beam can also be accommodated in the original analysis if so desired.

Reference 1 does not imply that out-of-plane effects are nonexistent. The study was restricted to the presentation of results for in-plane beam deformations because their effects were considered to be most interesting and important. Considerations of out-of-plane beam deformations are also included in the more complete treatment in Ref. 2. However, Kulla's plot in Fig. 5 is new.

Equation (3) of Ref. 1 contains a typographical error and should be corrected to read

$$T_b=\frac{\Omega^2}{2}\int_0^\ell \gamma A\Big\{2u^2(d)+2v^2(d)-\tfrac{1}{2}(\ell^2-y^2)\times\Big[\Big(\frac{\partial\eta_1}{\partial y}\Big)^2+\Big(\frac{\partial\eta_2}{\partial y}\Big)^2\Big]-2y^2\theta_1^2-2y(\eta_1+\eta_2)\theta_1\Big\}\mathrm{d}y$$

This correction does not affect any previously reported results.

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